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A simple model for the prediction of ground states of interacting 2D electron droplets and quantum dots

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Abstract. A simple model is presented for predicting the numerically 'observed' sequence of ground states (GSS) of 2D electron droplets and quantum dots in strong magnetic fields. The model is based on the assumption that exchange and correlation in the 2D electron gas induce gaps in the lowest Landau level, producing a sub-Landau level (SLL) structure. The sequence of GSS results from filling the SLLs as the total angular momentum L is increased. Using the numerically 'known' energies (for a few L) for $N \leqslant 6$ we present a fit formula to predict, to within 2% accuracy, the GSS for higher N and L for which exact diagonalization results are unavailable. Typical results for $N = 6$, 7 and 10 droplets are presented.

1. Introduction

The physics of interacting 2D finite N-electron systems (quantum dots, droplets) has been the object of recent experimental and theoretical studies (see e.g. Thornton *et al* 1986, Merkt 1990, McEuen *et al* 1991, Kastner 1992, Chakraborty 1992, Maksym 1993, Dharma-wardana *et al* 1992, Yang *et al* 1993). Analytical and numerical techniques have provided results for the ground state (GS) energy $E(L)$ of small disc-shaped N-electron systems, viz., quantum dots or droplets, as a function of the total angular momentum L. Laughlin's early study (Laughlin 1983) of a three-electron system considered a 2D electron droplet as well as its quadratic confinement via a 'pressure' term and essentially laid the groundwork for studies on quantum dots. The work of Girvin and Jach (1983), Maksym and Chakraborty (1992), Merkt (1990), Hawrylak (1993) and others, and the recent work of Yang *et al* (1993) for $N = 6$ for total angular momenta L going up to 45 for quadratically confined electrons, provide 'benchmark' results for the interpretation of experiments and for understanding the physics of correlated electrons. However, calculations for larger values of N, or larger L, become numerically prohibitive as the number of Slater determinants to be included increases very rapidly with N and L. Thus simple electrostatic models, constant-interaction (e.g., Coulomb blockade) 'charging' models, etc are often used in interpreting experiments and simply ignore the quantum physics of the problem. A more satisfying approach, which exploits the simplifications found in certain limiting situations for predicting the GS for a given N, L and B, and tailored to the results of exact quantum calculations, could often be more useful in this context. The objective of this paper is to provide just such a 'pocket calculator' model, for the strong-field, large-L limit, which corresponds to filling factors $\nu \leqslant 1$, constructed to obtain agreement with existing 'benchmark' calculations based on more fundamental approaches. After establishing the consistency of our model with existing results, we make predictive calculations for $N = 7$, 10-electron droplets for which numerical results are still unavailable. We also extend the range of energies $E(L)$ for $N = 5$, 6 beyond the L values so far available from exact diagonalizations.

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2. Basic ideas behind the model and its regime of applicability

Non-interacting 2D electrons in a field B and confined by a potential of the form $\frac{1}{2}m^*\omega_0^2 r^2$ can be treated exactly, giving single-particle states (n, ℓ) where n relates to the Landau level index in the high-field limit. When the Coulomb interaction is included, the many-body GSs can still be labelled using the total angular momentum $L = \sum_i \ell_i$ where ℓ_i is the angular momentum of the ith electron. The numerical diagonalizations provide a sequence of GSs as a function of L, which is optimal for the given confinement energy, Coulomb interaction and the field B. However, exact results for even moderate N, while clearly necessary for the interpretation of experiments, remain computationally prohibitive. This has led to the use of single-electron models, electrostatic 'charging' models, or models using a few interacting electrons even when N is not small. The objective of this paper is to use a few simple assumptions and construct a simple model, which correctly predicts the main sequence of many-body GSs as a function of L, for N-electron systems *confined to the lowest Landau level*. The model contains two fit parameters at three selected cusp points, which are fitted to reproduce the available 'benchmark' data for $N \leqslant 6$. A one-parameter fit to the GS energy of the 2D gas in the FQHE regime, using similar assumptions, is given by Dharma-wardana (1995).

We study the high-field spin-polarized, $T = 0$ K limit where the many-body GS energies separate into a confinement term $E_c(L)$ and an interaction term $E_i(L)$ with $E_c(L) = \hbar\omega_c N/2 + \gamma(L + N)$ where $\omega_c = eB/mc$. Here $\gamma = m^*\omega_0^2 a_L$, with $a_L^2 = \hbar c/eB$ following Yang *et al* (1993). The confinement energy increases linearly with L, but $E_i(L)$ is discontinuous and decreases with L, since the Coulomb repulsion favours the 'spreading out' of the wavefunction associated with increasing L. The interaction energy $E_i(L)$ in a quantum dot in a field B can be related to $E_i(L)$ without confinement at a field $B_0 = \Omega m^* c/e$ where $\Omega = (4\omega_0^2 + \omega_c^2)^{1/2}$. We assume that B has been mapped onto B_0 and simply use the symbol B for the effective 'external' magnetic field. Thus, in this approximation, all we need to study is the GS of the N-electron droplet in the effective field. Our objective is to predict $E_i(L)$ as a function of L, for arbitrary L and N, since $E_c(L)$ is simply $\gamma(L + N)$.

Plots of $E_i(L)$ obtained by numerical diagonalizations (Yang *et al* 1993) for $N = 5$ and 6 are given in figure 1, as a function of the total angular momentum L, scaled by $L_{min} = N(N - 1)/2$. Discontinuities in E_i with particle number or quantum number (e.g., L) are common in the physics of clusters where the filling of discrete energy shells ('incompressible' states) occurs. The average filling factor ν of a quantum droplet containing N electrons and total angular momentum L is given by $\nu = L_{min}/L$. Laughlin, Girvin and Jach, and subsequent authors, have speculated that the downward cusps (DCs) in $E_i(L)$ are in some way related to incipient quantum Hall effect- (QHE-) like condensations at filling factors ν corresponding to the odd-denominator filling factors of the fractional quantum Hall effect (FQHE). However, strong DCs are found at values of ν with even denominators, and at ν where no stable FQHE states are expected. Hence no clear correlation, except for the suggestion that DCs usually occur for $\Delta L = N$, was established. The $\Delta L = N$ rule does not usually give magic filling factors ν that agree with FQHE fractions. Maksym (1993) made a careful study of the occurrence of 'magic' L values where DCs occur and concluded that if $L = L_{min} + kN$, where k is an integer, then a stable GS (DC) occurs and the electrons are disposed at the corners of a regular N-sided polygon whose radius is equal to the classical circular orbit radius. However, DCs are also found at values of L that do not satisfy the above $\Delta L = N$ rule or agree with an FQHE value of ν. Here we show that the basic idea of FQHE-like condensation, when applied within the concept of the existence of a sub-Landau level (SLL) structure in the lowest Landau level (LLL), correctly predicts the numerically observed cusp structure using the *familiar* FQHE fractions.

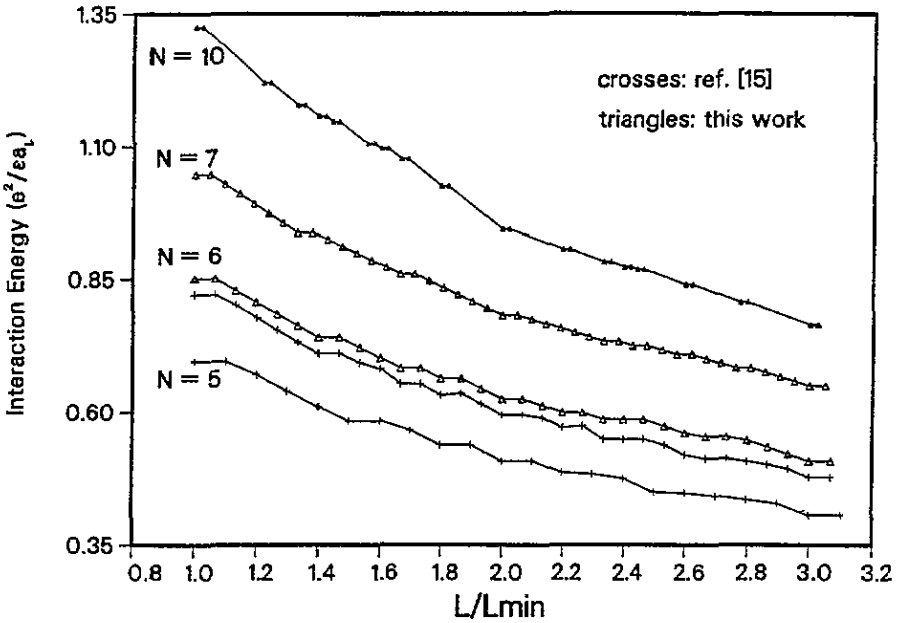

Figure 1. The interaction energy (in units of $e^2/\varepsilon a_L$) per electron as a function of $L/L_{\min}$, with $L_{\min} = N(N-1)/2$, for $N = 5$, 6, 7 and 10. The calculated curves (triangles) for $N = 6$ and $N = 7$ have been displaced upwards by 0.03 and 0.1 for clarity.

The essential idea is that, just as the integer QHE is based on filled Landau levels, the FQHE is based on the existence of an SLL structure (i.e. energy gap structure) in the LLL (Dharma-wardana 1991, 1992, 1995). The splitting of the LLL into SLLs occurs at the FQHE filling factors as a consequence of the electron–electron interaction. Thus our *ansatz* is that at $\nu = 1/m$, where m is an odd integer, the LLL is split into m SLLs (see appendix A). Some evidence for such structure is seen in numerical studies of 2D electron systems in magnetic fields (e.g., see Rejaei 1993), and perhaps also in the surprisingly large oscillatory 'shell structure' found (Mitra and MacDonald 1993) in the angular momentum state occupation number distributions of Laughlin droplets. The SLLs themselves can be further split into sub-sub-Landau levels (i.e. a higher level of energy gap structure), as seen in the hierarchy scheme (Haldane 1983, MacDonald *et al* 1985) of the FQHE. Instead of using an SLL concept, the composite fermion approach of Jain (1989) uses a hierarchy of integer quantum Hall states to map the fractional quantum Hall fluids. More remarks about the composite fermion approach are given in appendix B.

Thus our model is based on looking at magic L values in the LLL and in the SLLs, keeping in mind that in *finite* N-electron systems, edge effects not found in the infinite fluid come into play.

3. Prediction of 'magic' L values for N-electron droplets and quantum dots

We consider the high-field limit interaction energy E_i of the N-electron droplet. The lowest possible angular momentum state for N electrons has $L = L_{\min} = N(N-1)/2$, with the electrons occupying the LLL ($n = 0$) and angular momentum states $\ell = 0, 1, \ldots, N-1$. The

wavefunction is a Slater determinant (SLD) with filling factor $\nu = L/L_{\min} = 1$. Consider the case $L = 3L_{\min}$. The nominal filling factor $\nu = \frac{1}{3}$. Alternatively, we picture this as consisting of three SLLs, each full, using renormalized particles with charge $\frac{1}{3}$ (or particles of unit charge with a renormalized effective magnetic field $B^* = B/3$ (see appendix A)). If we denote the filling factors of SLLs labelled s by ν_s^*, then, when $L = kL_{\min}$, we have enough states to form k SLLs, each of which is full, i.e. with effective filling factors $\nu_s^* = 1$, $s = 1, 2, \ldots, k$. That is, the nominal filling factor $1/k$ is interpreted as the complete filling of k SLLs. Completely filled SLLs correspond to specially stable GSs and hence give rise to DCs. There is no difficulty with *even* values of k: their special stability should disappear as the number of electrons N tends to infinity. The case $k = 2$ has been described by a gapless Fermi-liquid-like theory (Halperin *et al* 1993) for the large-N limit of a uniform 2D electron gas. We discuss the case $k = 2$ for finite N below.

Consider an L such that $L_{\min} < L < 2L_{\min}$. There are enough L states to fill one SLL but the second SLL will be only partially filled. Since all quasielectrons participate equally in the SLLs, we must have $L - L_{\min} \geqslant N$ to have an effective SLL. Thus $L = L_{\min} + N$ corresponds to the case where the additional increment in L over $L_{\min}$ is just sufficient to *provide one flux quantum (one vortex) per electron in the new* SLL. *Hence* $L = L_{\min} + N$ *is* expected to be a DC. This case also corresponds to the $\Delta L = N$ rule discussed by Laughlin and also examined by Maksym. When we go from $L_{\min}$ to $L_{\min} + 1$ the interaction energy does not change. Then, until $L = L_{\min} + N$ the interaction energy E_i decreases essentially *quasilinearly with* L as each flux quantum (vortex) corresponding to each increment of L binds to an electron and increases the screening. Thus the first important DC occurs at $\nu_2^* = N/L_{\min}$. This corresponds to the nominal filling factor $\nu = (N-1)/(N+1)$ and is *not* necessarily a familiar FQHE fraction.

Suppose $L_{\min} + N < L < 2L_{\min}$. The first SLL is filled (i.e. $\nu_1^* = 1$) and the second SLL has $\nu_2^* = (L - L_{\min})/L_{\min}$. If ν_2^* is of the form p/q where q is an odd integer, then ν_2^* will also define a DC and the second SLL acquires its own sub-SLL structure similar to the hierarchy structure found in an FQH liquid. Unlike in the infinite 2D electron gas, we assume (see below) that the two-SLL case, $L = 2L_{\min}$, $\nu = \frac{1}{2}$, i.e. $\nu_1^* = 1$, $\nu_2^* = 1$, defines states separated by an energy gap (for finite N, due to the edge energy) and this would correspond to a DC, with a wavefunction approximately corresponding to a product of two functions, viz., that corresponding to SLLs ν_1^* and ν_2^*. In the case of a three-electron droplet, $\nu = \frac{1}{2}$ is obtained for $L = 6$, with $L_{\min} = 3$. The analytic many-body wavefunction (Laughlin 1983) can in fact be written in an approximate product form as

$$\Psi(\nu_1^* = \nu_2^* = 1) = \{z_a(3z_a^2 - z_b^2)\exp(-|z_a|^2/4)\}\{z_b(3z_b^2 - z_a^2)\exp(-|z_b|^2/4)\} \tag{1}$$

where z_a and z_b are given in terms of the two relative coordinates z_1 and z_2 defined relative to the centre of mass of the three-particle system, viz., $z_a = (\frac{3}{2})^{1/2}(z_1 + z_2)$ and $z_b = (\frac{1}{2})^{1/2}(z_1 - z_2)$. The GSs of N-particle droplets, for small N, resemble small crystallites, which become liquid-like for large N. While $\nu = \frac{1}{2}$ requires a careful description for large N, a 'Wigner molecule'- (Maksym 1993) type description is probably appropriate for $\nu = \frac{1}{2}$ at finite N.

Our discussion implies that DCs do not depend on $\nu = L_{\min}/L$, but on $(L - kL_{\min})/L_{\min}$ where k defines the last filled SLL with $\nu_k^* = 1$. Since $L_{\min} = N(N-1)/2$, different N will emphasize different FQHE p/q fractions. In table 1 we give the predicted GSs at DCs *for* $N = 6$, 7 *and* 10, *based on the assumption that for a given nominal filling factor* ν, *if* the SLL filling ν_k^* is unity or an FQHE p/q, then we have a DC (the $\Delta L = N$ rule already mentioned is used only for the first DC after $L = L_{\min}$, other remarks are given below).

The cusps predicted using these ideas agree with published $N \leqslant 6$ exact diagonalization results. Thus Yang *et al* (1993) report $L = 15, 21, 25, 30, 35, 39, 40$ and 45 for the GS of the fully spin polarized $N = 6$ quantum dot, *inclusive* of the confinement energy. A direct examination of *only* the interaction energy (Yang 1994) in the high-field limit for $N = 6$ shows DCs at $L = 15$, (19, 20), 21, 25, 27, 30, 33, 35, (36), 39, 40 and 45, where the parentheses indicate weaker DCs. For $N = 5$ Yang *et al* obtain (Yang 1994) DCs at $L = 10$, (14), 15, 18, 20, 22, 25, (26) and 30. All the main DCs are picked up by our 'aufbau' procedure.

Table 1. The 'magic' values of the total angular momentum, giving favoured GSs (DCs) for $N = 6$, 7 and 10, predicted using the model of SLL filling, in the high-field limit for nominal filling factors $\frac{1}{3} \leqslant \nu \leqslant 1$. For $N = 10$ DCs should also occur at $L = 65$, 70 and 81 for $\nu_2^* = \frac{4}{9}$, $\frac{5}{9}$ and $\frac{4}{5}$ and $L = 99$, 100, 105 and 110 for $\nu_3^* = \frac{1}{5}$, $\frac{2}{9}$, $\frac{3}{7}$ and $\frac{4}{9}$. These are in figure 1 but *not* listed below for brevity. We also give the nominal filling factor $\nu = L_{\min}/L$ for $N = 10$. The fractions marked with an asterisk correspond to $\nu = (N-1)/(N+1)$.

$N = 6$		$N = 7$		$N = 10$		
L	$\{\nu_s^*\}$	L	$\{\nu_s^*\}$	L	$\{\nu_s^*\}$	ν
15	1	21	1	45	1	1
21	$1, \frac{2}{5}^*$	28	$1, \frac{1}{3}^*$	55	$1, \frac{2}{9}^*$	$\frac{9}{11}$
25	$1, \frac{2}{3}$	35	$1, \frac{2}{3}$	60	$1, \frac{1}{3}$	$\frac{3}{4}$
27	$1, \frac{4}{5}$	42	1, 1	63	$1, \frac{2}{5}$	$\frac{5}{7}$
30	1, 1	49	$1, 1, \frac{1}{3}$	72	$1, \frac{3}{5}$	$\frac{5}{8}$
33	$1, 1, \frac{1}{5}$	51	$1, 1, \frac{3}{7}$	75	$1, \frac{2}{3}$	$\frac{3}{8}$
35	$1, 1, \frac{1}{3}$	54	$1, 1, \frac{4}{7}$	90	1, 1	$\frac{1}{2}$
36	$1, 1, \frac{2}{5}$	58	$1, 1, \frac{2}{3}$	108	$1, 1, \frac{2}{5}$	$\frac{5}{12}$
39	$1, 1, \frac{3}{5}$	63	1, 1, 1	117	$1, 1, \frac{3}{5}$	$\frac{5}{13}$
40	$1, 1, \frac{2}{3}$			125	$1, 1, \frac{2}{3}$	$\frac{9}{25}$
45	1, 1, 1			135	1, 1, 1	$\frac{1}{3}$

As already stated, our 'aufbau' principle is based mainly on applying ideas valid for the bulk FQHE liquid (i.e. large-N limit) to droplets (i.e. finite N); as in the bulk FQHE fractions with large denominators should be excluded. For N-electron systems we retain the familiar FQHE, fractions p/q, but with $q \leqslant N - 1$. The $N - 1$ denominator is admitted only if the phase space (defined by L) is large enough, i.e., e.g., for filling the third SLL.

The discussion given so far enables us to predict the values of L for which DCs will occur. Given that this is a simple 'first-order' theory, there could be some ambiguities as we do not fully understand how edge effects and 'bulk effects' compete. That is, we should expect that we can predict the main DCs, but some cusp structures that arise from a subtle interplay of several effects would remain beyond the resolution of the present approach. We also need a method for estimating the *magnitude* $E_i(L)$ of the interaction energies without resort to detailed computation. This will be taken up in the next section. Plots of $E_i(L)$ as a function of $L/L_{\min}$ are given in figure 1, based on the method indicated in the following section.

4. Calculation of interaction energies by fitting to three 'benchmarks'

In comparing interaction energies $E(L, N)$ of N-electron droplets with total angular momentum L, we should compare them in their 'corresponding' filling factor states, viz.,

$E(\nu_1^*, \nu_2^*, \nu_3^*, \ldots; N)$ for different N. That is, electron droplets with the same SLL filling will form a sequence for fitting the interaction energy as a function of N. Thus for example, if we take $\nu = \frac{1}{3}$, i.e. $\nu_1^* = \nu_2^* = \nu_3^* = 1$, then $E_i(\nu = \frac{1}{3}, N)$ will form a sequence in N to which parameters could be fitted to extract the 'bulk' and 'edge' contributions to the $\frac{1}{3}$ droplet energy. The sequence $E_i(\nu = 1, N)$ would be another, different, sequence. There is a multitude of these sequences which, for $N \to \infty$, lead to the GS energies of the FQHE fluids at each ν. However, we use *only three of these sequences* to construct *all* the cusp sequences as a function of N. That is, we use the results of $N = 5$ and 6 from Yang *et al* (Yang 1994) for three 'benchmark' energies, viz., $E(L_{\min})$, $E(L_{\min} + N)$ and $E(2L_{\min})$, to 'bootstrap' for all other L and N. Here $E(L_{min})$ defines the interaction energy of the $\nu = 1$, viz., full single determinant. The DC at $E(L_{\min} + N)$ is a genuine finite-N-dependent effect. Similarly, $\nu = \frac{1}{2}$, i.e. $\nu_1^* = \nu_2^* = 1$, for finite N is more akin to two independent Landau levels rather than to the subtle Fermi liquid of Halperin *et al* (1993). Since the energy will consist of an 'edge energy' proportional to N, and a bulk term varying as N^2, we let

$$E(\nu_s^*; N) = a(\nu_s^*)N + b(\nu_s^*)N^2 \tag{2}$$

and obtain

$$E(L_{min}) = 0.065\,268N + 0.125\,986N^2 \tag{3}$$

$$E(L_{\min} + N) = -0.051\,433\,1N + 0.127\,036N^2 \tag{4}$$

$$E(2L_{\min}) = 0.071\,345\,0N + 0.087\,21N^2 \tag{5}$$

by fitting to results of Yang *et al* (1994). We also have, $E(L_{\min} + 1) = E(L_{\min})$ for arbitrary N. It appears that if the angular momentum $L_d = \{\nu_i^*\}$ defines a DC, then $E(L_d + 1) \approx E(L_d)$, suggesting that some form of HVZ-like theorem (Avron *et al* 1978) applies, even approximately, at each L_d. The energies from $L_{\min} + 1$ to $L_{\min} + N$ are assumed to be *a quasilinear decrease* in L. The physical picture is that, as additional flux quanta (vortices, i.e. density defects or exchange–correlation holes) are created by increasing L, they bind to electrons and contribute to the screening, thus decreasing the energy.

(2) is of the form $E(L_d; N) = a'(L_d)\nu^{1/2} + b'(L_d)\nu$ where ν is the bulk filling factor. We have pointed out (Dharma-wardana 1995) that when we go from a $\nu = 1$ fluid to a $\nu = 1/m$ FQHE liquid, the energy scales as $\nu^{1/2}$. Hence, using this '$\nu^{1/2}$ scaling *ansatz*' the fitting coefficients for arbitrary FQHE fractions can be predicted to be given by

$$a(1/m) = 0.065\,268(1/m)^{1/4} \qquad b(1/m) = 0.125\,986(1/m)^{1/2}. \tag{6}$$

Thus we can predict that $a(1/m) = 0.0496$, $b(\frac{1}{3}) = 0.0727$ while actual fits to the data of Yang *et al* (Yang 1994) at $L = 3L_{\min}$ give

$$E(3L_{\min}) = 0.048\,098\,7N + 0.071\,492\,7N^2. \tag{7}$$

Thus, judging from this case, the $\nu^{1/2}$ scaling *ansatz* gives energies to better than 2% accuracy, and provides coefficients a and b at other filling factors (e.g., $\nu = \frac{1}{5}$, i.e. $L = 5L_{\min}$, $\nu = \frac{1}{7}$ etc) where microscopic results are lacking. The success of the $\nu^{1/2}$ scaling *ansatz* shown by Dharma-wardana (1995) suggests that the 2% accuracy obtained here is probably typical. Using the $\nu^{1/2}$ scaling *ansatz* (6) the energies $E(L = 75)$ at $\nu = \frac{1}{5}$ and $E(L = 105)$ at $\nu = \frac{1}{7}$ for $N = 6$ are predicted to be 0.3817 and 0.3258 units of $e^2/\varepsilon a_L$

per electron. DCs for $45 \leqslant L \leqslant 105$ are predicted at $L = 45$, 48, 50, 51, 54, 55, 60, 63, 65, 66, 69, 70, 75, 78, 80, 81, 84, 85, 90, 93, 95, 96, 99, 100 and 105. No exact diagonalization results are currently available in this regime.

In summary, (6) extends available numerical $E(L, N)$ data to higher L and N values. In constructing the function $E(L, N)$ for arbitrary N we use equations (3), (4) and (5) together with the $\nu^{1/2}$ scaling of (6) when numerical 'benchmarks' are lacking. The interaction energies calculated by this procedure for $N = 6$, 7 and 10 are shown in figure 1.

5. Quality of the fit for $N < 5$

In figure 1 we show, e.g., for $N = 6$, data points for 32 filling factors, of which it is sufficient to fit just three values, e.g., $\nu = 1$, $\nu = (N-1)/(N+1)$ and $\nu = 2$, to reproduce exact diagonalization results. We have actually included the case $\nu = 3$, i.e. (7), as well to increase our accuracy, rather than using the $\nu^{1/2}$ scaling *ansatz* of (6).

How good is the parametrization for reproducing the energies for $N < 5$? Since our model is based on the mapping

$$\text{quantum dot} \rightarrow \text{quantum droplet} \rightarrow \text{uniform 2D gas}$$

with the assumption that the bulk terms dominate the edge effects, it is to be expected that the results would be poor for small N. Even in the case $N = 6$, if the six electrons are placed at the vertices of a hexagon, there are six internal interactions and six edge interactions along the perimeter. However, the situation seems to be quite analogous to that of the local density approximation of density functional theory, or Thomas–Fermi theory where an N-electron atom is modeled by a suitable jellium approximation. In our case we have used some inputs from the $N = 5$ and 6 cases. Thus the two-, three- and four-electron cases would correspond to treating the He, Li and Be atoms using some results for the energy of B and C extrapolated to the jellium model. It turns out that the parameterizations given by (2)–(7) significantly overestimate the interaction energies for the case $N = 2$, but work quite well for $N = 3$ and 4. In the $N = 2$ case, there is only a single interaction and hence the parameterization in terms of 'edge' and bulk interactions, implicit in (2) becomes invalid. If we consider the case $N = 4$, (3), (5) and (7) predict $E(L, N = 4)$ for $L = 6$, 12 and 18 to be 0.569, 0.420 and 0.334 (in units of $e^2/\varepsilon a_L$ per electron) respectively compared with the values 0.55, 0.42 and 0.32 of Girvin and Jach (1983). For $N = 3$, $L = 3$, 6 and 9, the fit formulae predict the energies 0.443, 0.333 and 0.262, compared with 0.4, 0.29 and 0.24 visually estimated from the figure of Girvin and Jach (1983). However, our fit model is irrelevant for small N where results of exact calculations are available.

6. Conclusion

We have presented a simple 'pocket calculator' model for the determination of the interaction energy of a 2D droplet of electrons in a *strong magnetic field* to within 2% accuracy for large N and L. The interaction energy can now be simply combined with the confinement energy and approximate results for the GS energy of electrons in nanostructures can be easily obtained, even for N and ν well beyond those accessible by exact diagonalization methods. The method 'bootstraps' form, e.g., three values of the numerically observed GS energies at nominal filling factors $\nu = 1$, $(N-1)/(N+1)$ and $\frac{1}{2}$, supplemented by a number of

theoretical ideas regarding the uniform fluid in the FQHE regime. The main ideas are that (i) energy gaps are created in the LLL, generating an SLL structure, due to electron–electron XC effects, (ii) DCs in energy, i.e. stable GSs, appear at familiar FQHE-like filling factors referred to these SLLs and (iii) increase of L *between* DCs corresponds to the creation of vortices (flux quanta), which bind to electrons and reduce the Coulomb interactions in a *quasilinear* manner. The *strong-field* GSs of quantum dots with a central antidot can also be treated by including the edge energies for an outer edge and an inner edge, using the functional form of the edge energy from (3)–(6). These ideas can eventually be extended to quantum dots in the partially spin-polarized, weak-field regimes, but this is outside the scope of this paper.

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Appendix A

As far as the present work is concerned, the proposed SLL structure really requires that the LLL has energy gaps that could be relabelled as 'subLL' gaps. However, as a matter of wider interest we point out (Dharma-wardana 1992, 1995) that the Laughlin wavefunction for a filling factor $\nu = 1/m$ can be written in the form of a product of m SLDs:

$$\psi(\nu = 1/m) = \prod_{s=1,\ldots,m} \phi_s(z_{1s}, z_{2s}, \ldots, z_{Ns})\delta(z_{1s}, z_t) \tag{A1}$$

$$\phi_s(z_1, z_2, \ldots, z_N) = \prod_{i<j}(z_{is} - z_{js}) \exp\left(-\sum_i |z_{is}|^2/a_L^{*2}\right).$$

Here $a_L^{*2} = ma_L^2$ is an effective magnetic length at an effective magnetic field $B^* = B/m$. When $m = 3$ we have three species ($s = 1, 2, 3$) and three SLDs $\Phi_s(z_{1s}, z_{2s}, \ldots, z_{Ns})$, each containing N particles, with one particle of each species at each coordinate z_{is}. In Laughlin's wavefunction there are m particles bound together at the coordinate z_i, i.e. as imposed by $\delta(z_{is}, z_{is'})$. A more general form would use a correlation function $g(z_{is}, z_{ts'})$ for $\delta(z_{is}, z_{is'})$. Thus the $\delta(z_{is}, z_{is'})$ condition may be viewed as a 'mean-field' approximation inherent in the Laughlin form. In this picture the electron charge is distributed over m components, and the electron is replaced by a quasielectron carrying a single vortex. The filling up of energy states in the SLLs is an 'aufbau' process using these quasielectrons. Since the effective field B^* is B/m, the effective filling factor $\nu^* = 1$ and (A1) describes m 'full' SLLs of *renormalized particles*.

Appendix B

In this appendix we make some, necessarily brief, remarks about the composite fermion method (Jain 1989). Its application to the prediction of GS energies of quantum dots (Jain

and Kawamura 1994) was brought to our attention in the course of the preparation of this paper. As discussed by Dharma-wardana (1992), our approach to the FQHE states of the 2D electron gas was suggested by density functional theory (DFT) ideas. In DFT the exchange and correlation effects produce an exchange–correlation gauge field $A_{xc}(\nu)$ whose effect is (in our view) equivalent to a Chern–Simons field in creating an effective field $B^* = B/m$ wherein the renormalized electrons have a filling factor $\nu^* = 1$. However, as far as this paper is concerned, the assumptions needed are (i) that the LLL is split into m SLLs for the case $\nu = 1/m$, (ii) that the filling up of these levels by renormalized particles occurs to give specially stable filling factors corresponding to filling individual SLLs (as in the integer quantum Hall effect) and partial fillings (as in the FQHE) of SLLs and (iii) that information from, say, a few special filling factors for $N = 5$ and 6 can be used to predict all other energies and energy cusps for other N and L. As long as these are treated as assumptions, ours is a 'model' rather than a 'first-principles theory' of electron energies in quantum dots and FQHE liquids.

In the composite fermion approach the LLL is not split into SLLs, but higher Landau levels are actually invoked. The electrons distribute among several Landau levels while binding an even number of flux quanta to become essentially non-interacting 'composite fermions'. The higher Landau levels are at energy scales involving multiples of $\hbar\omega_c$, that is, at very high energies in the limit of $B \to \infty$ considered by Jain and Kawamura. These are then supposed to map into quasi-Landau levels of composite fermions with an effective gap $\hbar\omega_c^m$, due to the binding of $2m$ flux quanta per electron. Although there seem to be *prima facie* similarities between the composite fermion description and our description motivated by DFT, we have not been able to establish a rigorous equivalence. However, it is pertinent to remark that while DFT describes the GS of interacting systems as simple products of SLDs, with the exchange–correlation effects included via the potentials V_{xc} and A_{xc} of current DFT (Vignale and Rasolt 1988), composite fermion theory includes exchange–correlation effects via Jastrow functions, which multiply the Slater determinants. In DFT, the 'Kohn–Sham eigenfunctions' are not considered to be wavefunctions, while the composite fermion approach claims to actually provide many-body wavefunctions. The situation is probably analogous to the Hylleraas description of the He atom using an SLD multiplied by a Jastrow-type factor, while the density functional treatment of He merely uses a single SLD, but with a $V_{xc}[n(r)]$ added to the Hartree Hamiltonian, where $n(r)$ is the electron density profile of the He atom. The two descriptions both use an SLD to describe fermions, but beyond that the methods are not easily translated from one to the other.

Jain and Kawamura (1994) introduce a kinetic energy mapping from higher Landau levels to the effective angular momentum L^* of non-interacting composite fermions and then an *ansatz* for the composite fermion interaction energy is also introduced. Now, using exact diagonalization results for $N \leqslant 6$ and fitting the effective cyclotron energies, these authors show that their assumptions are consistent with the available exact diagonalization results (the DC at $L = 40$, for the case $N = 6$, does not seem to have been captured by these authors), thus further substantiating the composite fermion approach. It is not easy for an experimentalist to use their formalism to make similar calculations, or extend the results to higher L or N. In fact the authors have not gone beyond the $N = 6$ case in applying their method although some results for cusp sizes up to $N = 8$ have been given. In effect, their approach is directed more towards a demonstration of the validity of composite fermion concepts than to the derivation of a simple calculational model. By contrast, our approach, while being admittedly less buttressed by a microscopic theory, is presented as a 'pocket calculator model', which is meant for rapid but reliable calculations of cusp structure and

energies of quantum dots and droplets in the strong-field regime. It is also hoped that our results will stimulate further work on exact diagonalizations for larger values of N and L.

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